BENDING OF AN UNBOUNDED PLATE SUPPORTED BY AN ELASTIC HALF-SPACE WITH A MODULUS OF ELASTICITY VARYING WITH DEPTH

(IZGIB NEOGRANICHENNOI PLITY NA UPRUGOM POLUPROSTRANSTVE S PEREMENNYM PO GLUBINE MODULEM UPRUGOSTI)

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The problem of the impression of a punch into a non-homogeneous elastic half-space has been considered in the works of Korenev [1] and Mossakovskii [2]. In the present note there is given the solution of the problem on the bending of an unbounded thin plate lying on an elastic half-space, whose modulus of elasticity is a power function of the depth. The plane case of this problem, i.e. the deflection of a beam-plate is treated with considerable detail. By the method of taking the limit there is obtained in a new form the solution of the deflection of a beam lying on a homogeneous elastic half-space [3].

1. It is known [4] that for the case of a half-space with a modulus of elasticity changing with the depth according to the law $E = E_\nu z^\nu$, the vertical displacements $w(x, y)$ of the boundary points $z = 0$ of the half-space and the normal stresses $p(x, y)$ on the plane $z = 0$ are connected by the relation

$$w(x, y) = \frac{1}{\pi D_\nu} \iint\limits_S \frac{p(\xi, \eta)\, d\xi d\eta}{\sqrt{[(x-\xi)^2 + (y-\eta)^2]^{\nu+1}}}, \qquad D_\nu^* = \frac{\alpha}{E_\nu} \tag{1.1}$$

where α is a coefficient depending on ν, and where Poisson's ratio μ is taken from graphs or published tables [4].

Here, as well as in earlier works [1,2], it is assumed that $0 \leqslant \nu < 1$.

Let us suppose that an unbounded thin plate with cylindrical rigidity D is supported by a half-space with the above mentioned properties. Furthermore, let us assume that the given plate is subjected to a load of the form

$$q(x, y) = \delta(x) \cos \lambda y, \qquad \lambda \geqslant 0$$

where $\delta(x)$ is the impulse function describing an initially applied concentrated unit force. In this case the stress $p(x, y)$ under the plate will have an analogous form, i.e. $p(x, y) = p_\lambda(x) \cos \lambda y$.

Substituting the given expression for $p(x, y)$ into the right-hand side of (1.1) , after some elementary transformations we obtain,

$$w(x,y) = w_\lambda(x)\cos\lambda y, \quad w_\lambda(x) = \int_{-\infty}^{\infty} k(|x-\xi|)\, p_\lambda(\xi)\, d\xi \qquad (1.2)$$

where

$$k(\alpha) = \frac{1}{\pi D_\nu} \int_{-\infty}^{\infty} (a^2+t^2)^{-\frac{\nu+1}{2}} \cos\lambda t\, dt \qquad (1.3)$$

Here, no tangential [shear] interaction between the plate and the supporting medium is taken into consideration, but it is assumed that there exists a reciprocal connection between the plate and the half-space. This means that the deflections of the plate are the same as the displacements of the boundary points of the half-space.

In view of this, the function $w(x, y)$ must satisfy the well known biharmonic equation of the theory of thin plates, i.e. in the given case we have the following equation

$$D\left(\frac{d^2}{dx^2} - \lambda^2\right)^2 w_\lambda(x) = \delta(x) - p_\lambda(x) \qquad (1.4)$$

Substituting (1.2) into (1.4) we obtain

$$p_\lambda(x) + \left(\frac{d^2}{dx^2} - \lambda^2\right)^2 D \int_{-\infty}^{\infty} k(|x-\xi|)\, p_\lambda(\xi)\, d\xi = \delta(x) \qquad (1.5)$$

This integro-differential equation is easily solved by the method of operational calculus. For this purpose we make use of Fourier transformation. We introduce the notation

$$\frac{\nu+3}{2} = \frac{m}{n}, \qquad \frac{D_\nu \Gamma(^1/_2(1+\nu))}{2^{1-\nu} D\Gamma(^1/_2(1-\nu))} = \gamma = c^m \qquad (1.6)$$

The solutions of the equation (1.5) can be represented in the form

$$p_\lambda(x) = \frac{c^m}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-iux}\, du}{c^m + (u^2+\lambda^2)^{m/n}} \qquad (1.7)$$

In view of (1.3), the Fourier transform of the kernel of the equation considered is

$$K(u)=\int_{-\infty}^{\infty}k(\alpha)e^{i\alpha u}d\alpha=\frac{1}{\pi D_\nu}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\cos\lambda t\cos u\alpha}{(\alpha^2+t^2)^{1/2(\nu+1)}}dt\,d\alpha \tag{1.8}$$

Applying to the double integral (1.8) a transformation analogous to the one used on the integral (2.63) of the work [5] (p. 171), we find that

$$K(u)=c^{-m}D^{-1}\sqrt{(u^2+\lambda^2)^{\nu-1}}$$

Substituting (1.8) into (1.2), we obtain

$$w_\lambda(x)=\frac{1}{2\pi D}\int_{-\infty}^{\infty}\frac{\sqrt{(u^2+\lambda^2)^{\nu-1}}\,e^{-iux}}{c^m+(u^2+\lambda^2)^{m/n}}du \tag{1.9}$$

2. We shall transform the slowly converging integrals occurring in (1.7) and (1.9) into rapidly converging ones with the aid of contour integration.

For this purpose we determine the zeros and singular points of the function

$$F_\lambda(w)=c^m+(w^2+\lambda^2)^{m/n} \tag{2.1}$$

We shall assume that m and n are integers. This restriction will be removed later.

Obviously, the function $F_\lambda(w)$ has two branch points, $w=\pm i\lambda$. In order to find the zeros of the function (2.1), we represent it in the form

$$F_\lambda(w)=\prod_{k=1}^{m}\left(\sqrt[n]{w^2+\lambda^2}-a_k\right),\qquad a_k=c\cdot\exp\left(i\pi\frac{1-2k}{m}\right)\qquad (k=1,\ldots,m) \tag{2.2}$$

In what follows let us select that single-valued branch of the function

$$z=(w^2+\lambda^2)^{m/n}$$

which is determined in the plane with the excluded rays $(-i\infty,-i\lambda)$ and $(i\infty, i\lambda)$ and by means of which the indicated region is mapped on the wedge-shaped region

$$|\arg z|=\pi\frac{m}{n}\qquad\text{or}\qquad\left|\arg\sqrt[n]{(w^2+\lambda^2)^m}\right|<\frac{\pi m}{n}$$

and particularly, on

$$\left|\arg\sqrt[n]{w^2+\lambda^2}\right|<\frac{\pi}{n}$$

Hence, all zeros of the function $F_\lambda(w)$ will have to be numbers of the form $\sqrt{(a_k{}^n - \lambda^2}$, $k = 1, \ldots, m$, for which the following inequality holds

$$|\arg a_k| \leqslant \frac{\pi}{n} \tag{2.3}$$

Taking into account the fact that $m > n$, it is not difficult to show that the numbers

$$a_1 = c\exp\left(-i\frac{\pi}{m}\right) = b_2, \qquad a_m = c\exp i\frac{\pi}{m} = b_1 \tag{2.4}$$

satisfy the condition (2.3).

It is impossible to select from the set $a_1, \ldots, a_m$ any numbers, other than those given in (2.4), that could satisfy condition (2.3). This follows from the fact that the arguments of the numbers a_k decrease in a monotonic way from $\arg a_1 = -\pi/m$ to $\arg a_m = \pi/m - 2\pi$, and that even a_2 does not satisfy condition (2.3), because $3/2 < m/n < 2$ in view of (1.6) and of the inequality $0 \leqslant \nu < 1$; finally, the numbers a_k have to be pairs of conjugate numbers.

Thus, we see that in the general case $(0 \leqslant \nu < 1)$ the function $F_\lambda(w)$ will have only two zeros in the lower and only two in the upper half-planes

$$w = \alpha_j = \sqrt{b_j{}^n - \lambda^2}, \qquad \operatorname{Im}\sqrt{b_j{}^n - \lambda^2} > 0 \qquad (j = 1, 2) \tag{2.5}$$

$$w = -\alpha_j \qquad (j = 1, 2) \tag{2.6}$$

3. Let us return now to the formulas (1.7) and (1.9). Bearing in mind the considerations of Section 2, we now proceed to transform the path of integration for the integral occurring on the right-hand part of (1.7); for $x < 0$ the path becomes a loop containing the ray $(i\lambda, i\infty)$, for $x > 0$, a loop enclosing the ray $(-i\lambda, -i\infty)$.

Carrying out the suggested transformations, making use of (1.6) and of the notation $c^m = \gamma$, in place of (1.8) we obtain the following result

$$p_\lambda(x) = \gamma\left(\frac{1}{\pi}\sin\frac{\nu+3}{2}\pi\int_\lambda^\infty \frac{(s^2-\lambda^2)^{1/2(\nu+3)}\, e^{-s|x|}\, ds}{\gamma^2 + 2\gamma\cos\left[{}^1/_2(\nu+3)\pi\right](s^2-\lambda^2)^{1/2(\nu+3)} + (s^2-\lambda^2)^{\nu+3}} + \right.$$

$$\left. + \, i\sum_{j=1}^{2}\frac{1}{\nu+3}\frac{e^{i\alpha|x|}}{\alpha_j\beta_j{}^{m-n}}\right) \tag{3.1}$$

One can show without difficulty that $b_{1,2}{}^{m-n} = (b_{1,2}{}^n)^{(\nu+1)/2}$, and that

$$b_{1,2}{}^n = \gamma^{2/(\nu+3)}\exp\left(\pm i\pi\frac{2}{\nu+3}\right) \tag{3.2}$$

The formula (1.9) can be transformed in an analogous way to give

$$w_\lambda(x) = -\frac{\gamma}{\pi D}\sin\frac{\nu+3}{2}\pi\int_\lambda^\infty \frac{(s^2-\lambda^2)^{1/2(\nu-1)}e^{-s|x|}ds}{\gamma^2+2\gamma\cos[{}^1/_2(\nu+3)\pi](s^2-\lambda^2)^{1/2(\nu+3)}+(s^2-\lambda^2)^{\nu+3}} + \\ + i\sum_{j=1}^{2}\frac{1}{\nu+3}\frac{e^{i\alpha_j|x|}}{\alpha_j b_j{}^n} \tag{3.3}$$

The expressions for the stress $p(x, y)$ under the plate, and for the deflections of a plate loaded with a unit concentrated initial force will be

$$p(x, y) = \frac{1}{\pi}\int_0^\infty p_\lambda(x)\cos\lambda y d\lambda, \quad w(x, y) = \frac{1}{\pi}\int_0^\infty w_\lambda(x)\cos\lambda y d\lambda \tag{3.4}$$

If, herein, one has to use rectangular coordinates (for example, when one investigates the behavior of a foundation plate under a rectangular grid of columns) then one has to substitute $p_\lambda(x)$ and $w_\lambda(x)$ from (3.1) and (3.3) into the right-hand side of formula (3.4).

In those cases when polar coordinates are more suitable (under symmetric axial loading) one should select the original expressions (1.7) and (1.9) for $p_\lambda(x)$ and $w_\lambda(x)$, respectively.

For the stress $p(r)$ under the plate, subjected to the initial unit force, and for the displacement $w(r)$ we will then have the following expressions

$$p(r) = \frac{\gamma}{2\pi}\int_0^\infty \frac{\rho I_0(\rho r)}{\gamma+\rho^{\nu+3}}d\rho, \quad w(r) = \frac{1}{2\pi D}\int_0^\infty \frac{\rho^{(\nu)}I_0(\rho r)}{\gamma+\rho^{\nu+3}}d\rho \tag{3.5}$$

$$r = \sqrt{x^2+y^2}$$

4. Let us dwell upon the plane case of this problem in greater detail. First, we consider the problem of the deflection of a beam-plate under a unit load concentrated along the line $x = 0$.

It is not difficult to see that the stress $p_I(x)$ under the plate will be determined in this case by means of the limit relations

$$p_I(x) = \lim p_\lambda(x) \qquad \text{for } \lambda \to 0 \tag{4.1}$$

while the bending moment, $M_I(x) = Dy''(x)$, of the plate is given by

$$M_I(x) = \lim_{\lambda\to 0} D\frac{d^2 w_\lambda(x)}{dx^2} \tag{4.2}$$

Setting $\lambda = 0$ on the right-hand side of (3.1), and bearing in mind that on the basis of (2.5) and (3.2)

$$\lim_{\lambda\to 0}\alpha_{1,2}=\pm\gamma^{\frac{1}{\nu+3}}\exp\left(\pm\frac{\pi}{\nu+3}\right)$$

we obtain

$$p_I(x)=\gamma^{\frac{1}{\nu+3}}\left\{\frac{1}{\pi}\int_0^\infty\frac{\sin\left[{}^1/_2(\nu+3)\pi\right]s^{\nu+3}\exp\left(-\gamma^{\frac{1}{\nu+3}}s\,|\,x\,|\right)ds}{1+2\cos\left[{}^1/_2(\nu+3)\pi\right]s^{\nu+3}+s^{2(\nu+3)}}+\right. \tag{4.3}$$

$$\left.+\frac{2}{\nu+3}\sin\left(\gamma^{\frac{1}{\nu+3}}|\,x\,|\cos\frac{\pi}{\nu+3}+\frac{\pi}{\nu+3}\right)\exp\left(-\gamma^{\frac{1}{\nu+3}}|\,x\,|\sin\frac{\pi}{\nu+3}\right)\right\}$$

Analogously, after two differentiations, from (3.2) we derive the result

$$M_I(x)=-\gamma^{-\frac{1}{\nu+3}}\left\{\frac{1}{\pi}\int_0^\infty\frac{\sin\left[{}^1/_2(\nu+3)\pi\right]s^{\nu+1}\exp\left(-\gamma^{\frac{1}{\nu+3}}|\,x\,|\,s\right)ds}{1+2s^{\nu+3}\cos\left[{}^1/_2(\nu+3)\pi\right]+s^{2(\nu+3)}}-\right.$$

$$\left.-\frac{2}{\nu+3}\sin\left(\gamma^{\frac{1}{\nu+3}}|\,x\,|\cos\frac{\pi}{\nu+3}-\frac{\pi}{\nu+3}\right)\exp\left(-\gamma^{\frac{1}{\nu+3}}|\,x\,|\sin\frac{\pi}{\nu+3}\right)\right\} \tag{4.4}$$

Making use of the obtained expressions for $p_I(x)$ and $M_I(x)$, one can give quite simple formulas for finding the maximum stress, max p_I, under the plate and the maximum bending moment, max M_I, of the plate. These formulas can be written as

$$\max\begin{cases}p_I\\ M_I\end{cases}=\frac{1}{\nu+3}\gamma^{\pm\frac{1}{\nu+3}}\left(2\sin\frac{\pi}{\nu+3}+\frac{\cos\left[2\pi/(\nu+3)\right]}{\sin\left[\pi/(\nu+3)\right]}\right) \tag{4.5}$$

In order to obtain the formulas (4.5) from (4.3) and (4.4) one has to set $x = 0$. After that one must transform the resulting improper integrals to a form which permits the use of the following relation

$$\frac{1}{\pi}\int_0^\infty\frac{t^\alpha dt}{1-2t\cos\lambda+t^2}=\frac{\sin\alpha(\pi-\lambda)}{\sin\lambda\sin\pi\alpha},\qquad \lambda^2<\pi^2 \tag{4.6}$$

for their evaluation.

The last relation was obtained by the method of contour integration. Hereby the integration contour consisted of a loop surrounding the origin of the coordinate system and the interval (O, R) of the real axis, and of the circle of radius $R \to a$.

Setting in formulas (4.3) and (4.4)

$$\nu=0,\quad \gamma^{\frac{1}{\nu+3}}=\{E_0\left[2(1-\mu^2)D\right]^{-1}\}^{\frac{1}{3}}=c_0 \tag{4.7}$$

we find the following formulas for the stress $p_0(x)$ under the plate and for the bending moment $M_0(x)$ in the beam-plate lying on an elastic homogeneous half-space and subjected to a unit load concentrated along a line

$$p_0(x) = \frac{2c_0}{3} e^{-0.5\sqrt{3}c_0|x|} \cos\left(\frac{\pi}{6} - \frac{c_0|x|}{2}\right) - \frac{c_0}{\pi}\int_0^\infty \frac{e^{-c_0|x|s}\, s^3 ds}{1+s^6} \tag{4.8}$$

$$M_0(x) = -\frac{2}{3c_0} e^{-0.5\sqrt{3}c_0|x|} \cos\left(\frac{\pi}{6} + \frac{c_0|x|}{2}\right) + \frac{1}{\pi c_0}\int_0^\infty \frac{c^{-c_0|x|s}\, s ds}{1+s^6} \tag{4.9}$$

These formulas, which were first obtained by us in an earlier work [3], have certain advantages over formulas given by other authors; it should be noted that formula (10) of the earlier work [3] contains a misprint; in place of $\cos(\pi/2 - c|x|/2)$ it should be $\cos(\pi/6 - c|x|/2)$.

5. If along the line $x = 0$ of a beam-plate there is acting a concentrated turning moment, rotating in a clockwise direction, then the formulas of the bonding moment $M_{II}(x)$ along a plate cross-section, and the stress $p_{II}(x)$ under the plate can be obtained easily by starting out with the formulas (4.3) and (4.4) and the use of the relationship

$$M_{II},\ p_{II} = -\frac{d}{dx}(M_I,\ p_I) \tag{5.1}$$

In case the beam-plate is subjected to a unit deformation [5] along the section $x = 0$, one uses the relationship (compare [3])

$$M_{III},\ p_{III} = \frac{d^2}{dx^2}(M_I,\ p_I) \tag{5.2}$$

for the computation of the bending moment $M_{III}(x)$ of the plate and of the stress $p_{III}(x)$ under the plate.

In conclusion, we note that it is possible to obtain the exact solution of the deflection problem of a semi-infinite plate lying on an elastic half-space of the type considered here.

This can be done by a method analogous to the one used in [7]. First, one reduces the problem to an integral equation of the first kind and of the Wiener-Hopf type, after that one solves this equation by the use of a known procedure [6].

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